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Couette flow for a gas with a discrete velocity distribution

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We consider a kinetic theory model of a gas, whose molecular velocities are restricted to a set of fourteen given vectors. For this model we study the Couette flow problem, the boundary conditions on the walls being the conditions of pure diffuse reflexion. The kinetic equations can be integrated by quadrature under the assumption that the walls have opposite velocities and equal temperatures. The presence on the walls of tangential velocities leads to the consequence that the velocity slip coefficient does not in general vanish when the Knudsen number goes to zero.

Considering the same problem again after the suppression of tangential velocities, we obtain formulae for the velocity and temperature slip coefficients which generalize results of Broadwell (1964*b*), and which agree qualitatively with experiments.

1. Introduction

The study of flow in the kinetic theory of gases is a difficult problem which can be successfully treated only if one first replaces the Boltzmann equation by a model equation. Among the methods proposed to simplify the Boltzmann equation is the method of discretization of velocity space, as first suggested by Maxwell. This method consists of assuming that the molecules of the gas can have only certain discrete velocities chosen from a given set of p vectors. The Boltzmann equation is then replaced by a system of p quasi-linear partial differential equations. For different models, various particular problems have been solved: the structure of shock waves by Broadwell (1964*a*) and Gatignol (1975*a*, *b*), the Couette flow problem and Rayleigh problem by Broadwell (1964*b*), the approach to equilibrium by Harris (1966), thermodynamics and hydrodynamics for a model with four velocities by Hardy & Pomeau (1972). In all these studies except that of Harris, the velocities of molecules all have the same modulus; as a consequence the temperature is not a macroscopic independent variable, for it can be expressed as a function of the mean velocity. In particular the Couette flow problem studied by Broadwell concerns a gas with eight velocities, obtained by joining the centre of a cube to the vertices; the centre of the cube is at the origin of the velocity space, and the walls which bound the fluid are parallel to one of the faces of the cube. The simplest three-dimensional regular model in which the velocities do not have the same modulus is the model with fourteen velocities obtained by adding to these eight velocities six new vectors joining the centre of the cube to the centre of the faces (Cabannes 1975). The subject of this paper is first to consider the Couette flow problem on the basis of this model; this new study reveals the influence of the temperature on the different quantities and on the different parameters which characterize the flow; the study of the fractional slip velocity, in particular, leads for certain temperatures to negative values, and further this coefficient does not in general go to zero when the Knudsen number goes to zero (the continuous flow limit). These two paradoxical results are due to the existence of velocities parallel to the walls, whereas in reality such velocities constitute a set of zero measure. The paradoxes disappear if we remove the four velocities tangential to the walls, as is (non-trivially) possible, for after this removal there still exist mixed collisions, i.e. collisions between molecules having velocities of different moduli. In § 5 we consider the Couette flow problem on the basis of this model with ten velocities and obtain results which are in qualitative agreement with experiments.

2. Equations of the problem

The velocities are denoted by $\mathbf{u}_i$ ($i = 1, \ldots, 8$) and $\mathbf{v}_j$ ($j = 1, \ldots, 6$), and their components in the directions Ox, Oy, Oz of a Cartesian system are (figure 1)

$$\mathbf{u}_1 = c(-1,1,1) \quad \mathbf{u}_2 = c(1,1,1), \quad \mathbf{u}_3 = c(-1,-1,1), \quad \mathbf{u}_4 = c(1,-1,1),$$

$$\mathbf{v}_1 = c(1,0,0), \quad \mathbf{v}_2 = c(0,1,0), \quad \mathbf{v}_3 = c(0,0,1)$$

and $$\mathbf{u}_{9-i} = -\mathbf{u}_i \quad (i = 1,2,3,4), \quad \mathbf{v}_{j+3} = -\mathbf{v}_j \quad (j = 1,2,3),$$

the moduli being given by $|\mathbf{v}_j| = c, \quad |\mathbf{u}_1| = 3^{\frac{1}{2}}c.$

The number density of molecules with velocity $\mathbf{u}_i$ is denoted by N_i, that of molecules with velocity $\mathbf{v}_j$ by M_j. The kinetic equations as derived in an earlier paper (Cabannes 1975) are then

$$\begin{aligned}\frac{\partial N_1}{\partial t} - c\frac{\partial N_1}{\partial x} + c\frac{\partial N_1}{\partial y} + c\frac{\partial N_1}{\partial z} &= \tfrac{1}{2} \times 3^{\frac{1}{2}}cS(N_2N_7 + N_3N_6 + N_4N_5 - 3N_1N_8)\\ &+ 2^{\frac{1}{2}}cS(N_2N_3 + N_3N_5 + N_5N_2 - N_1N_4 - N_1N_6 - N_1N_7)\\ &+ \tfrac{1}{2} \times 6^{\frac{1}{2}}cS(N_2M_4 + N_3M_2 + N_5M_3 - N_1M_1 - N_1M_5 - N_1M_6), \quad (1)\end{aligned}$$

$$\begin{aligned}\frac{\partial M_1}{\partial t} + c\frac{\partial M_1}{\partial x} &= \tfrac{2}{3}cS(M_2M_5 + M_3M_6 - 2M_1M_4) + \tfrac{1}{2} \times 6^{\frac{1}{2}}cS(N_2M_4\\ &+ N_4M_4 + N_6M_4 + N_8M_4 - N_1M_1 - N_3M_1 - N_5M_1 - N_7M_1) \quad (2)\end{aligned}$$

together with seven equations similar to (1) and five equations similar to (2) obtained in an obvious way by permutation of suffixes. In these equations S is a constant representing the collision cross-section. These equations, fourteen in all, are satisfied, in kinetic theory, by the motions of a gas with fourteen velocities. When we assume (as we shall now do) that the distribution of velocities is symmetric with respect the Oxy plane, the number of unknowns and of equations is reduced to nine; the unknowns are the densities N_i ($i = 1, 2, 3, 4$) and the densities M_j ($j = 1, \ldots, 6$) except M_6, which is equal to M_3.

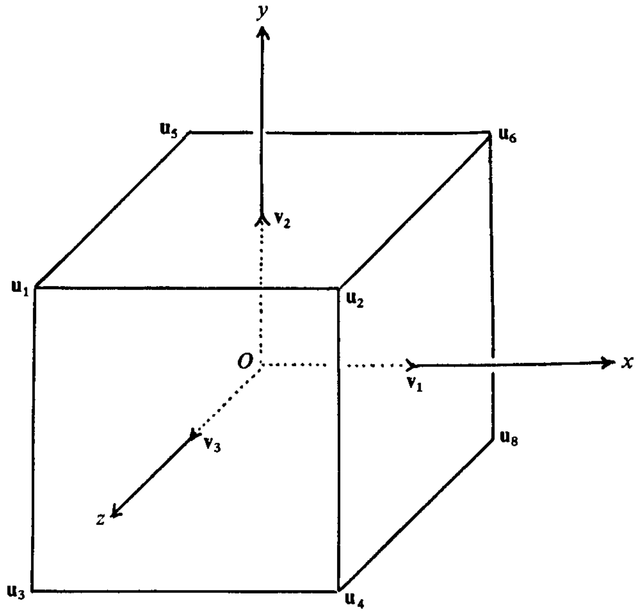

FIGURE 1. Model with fourteen velocities.

The Couette flow problem consists of determining the steady flow between two plane parallel walls $y = \pm d$ moving with constant velocities $(-U_w, 0, 0)$ and $(+U_w, 0, 0)$, where $U_w > 0$, and having the same constant temperature T_w. By analogy with the classical Couette flow problem we shall look for (and obtain) a solution of the problem under the assumption that the unknown functions (i.e. the densities) depend only on the variable y. The equations then take the form

$$dN_1/dy = (2^{\frac{1}{2}}+3^{\frac{1}{2}})S(N_2N_3-N_1N_4)+\tfrac{1}{2}\times 6^{\frac{1}{2}}S(N_2M_4+N_3M_2-N_1M_1-N_1M_5), \quad (3a)$$

$$dN_2/dy = (2^{\frac{1}{2}}+3^{\frac{1}{2}})S(N_1N_4-N_2N_3)+\tfrac{1}{2}\times 6^{\frac{1}{2}}S(M_1N_1+N_4M_2-N_2M_4-N_2M_5), \quad (3b)$$

$$dN_3/dy = (2^{\frac{1}{2}}+3^{\frac{1}{2}})S(N_2N_3-N_1N_4)+\tfrac{1}{2}\times 6^{\frac{1}{2}}S(N_3M_1+N_3M_2-N_1M_5-N_4M_4), \quad (3c)$$

$$dN_4/dy = (2^{\frac{1}{2}}+3^{\frac{1}{2}})S(N_1N_4-N_2N_3)+\tfrac{1}{2}\times 6^{\frac{1}{2}}S(N_4M_2+N_4M_4-N_2M_5-N_3M_1), \quad (3d)$$

$$0 = \tfrac{2}{3}S(M_2M_5+M_3^2-2M_1M_4)+6^{\frac{1}{2}}S(N_2M_4+N_4M_4-N_1M_1-N_3M_1), \quad (3e)$$

$$\frac{dM_2}{dy} = \tfrac{2}{3}S(M_1M_4+M_3^2-2M_2M_5)+6^{\frac{1}{2}}S(N_1M_5+N_2M_5-N_3M_2-N_4M_2), \quad (3f)$$

$$0 = \tfrac{2}{3}S(M_1M_4+M_2M_5-2M_3^2), \quad (3g)$$

$$0 = \tfrac{2}{3}S(M_2M_5+M_3^2-2M_1M_4)+6^{\frac{1}{2}}S(N_1M_1+N_3M_1-N_2M_4-N_4M_4), \quad (3h)$$

$$\frac{dM_5}{dy} = \tfrac{2}{3}S(2M_2M_5-M_1M_4-M_3^2)+6^{\frac{1}{2}}S(N_1M_5+N_2M_5-N_3M_2-N_4M_2). \quad (3i)$$

Equations ($3e$–i) can be replaced by the following:

$$\left.\begin{aligned} &M_1M_4 = M_2M_5 = M_3^2,\\ &(N_1+N_3)M_1 = (N_2+N_4)M_4,\\ &dM_2/dy = 6^{\frac{1}{2}}S(N_1M_5+N_2M_5-N_3M_2-N_4M_2),\\ &M_5-M_2 = H \quad \text{(constant)}.\end{aligned}\right\} \quad (4)$$

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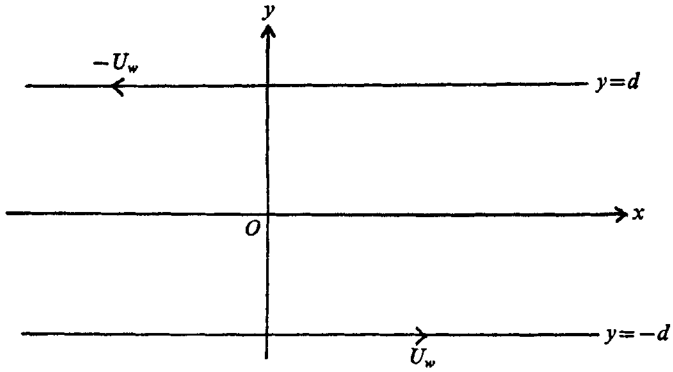

FIGURE 2. Couette flow.

From (4) and ($3a$–d) we deduce

$$\left.\begin{aligned} N_1 - N_3 &= K_1 \quad \text{(constant)}, \\ N_2 - N_4 &= K_2 \quad \text{(constant)}. \end{aligned}\right\} \tag{5}$$

The sum $c\{2(N_1+N_2-N_3-N_4)+M_2-M_5\}$ has the constant value

$$c\{2(K_1+K_2)-H\}$$

and represents the flux in the y direction.† The boundary conditions imply that this flux is zero on the walls and therefore zero everywhere, and we have $H = 2(K_1+K_2)$. This relation between the constants leads to

$$\left.\begin{aligned} dM_2/dy &= 6^{\frac{1}{2}}SH\{N_1+N_2+\tfrac{1}{2}M_2\}, \\ 2(dN_1/dy+dN_2/dy) &= -6^{\frac{1}{2}}SH\{N_1+N_2+\tfrac{1}{2}M_2\}. \end{aligned}\right\} \tag{6}$$

By addition we obtain a new first integral

$$2(N_1+N_2)+M_2 = 2K \quad \text{(constant)}, \tag{7}$$

or

$$2(N_3+N_4)+M_5 = 2K. \tag{7'}$$

The derivative dM_2/dy has the constant value $2\times 6^{\frac{1}{2}}SHK$, and we can write

$$M_2(y) = M_2(0) + 2\times 6^{\frac{1}{2}}SHKy, \tag{8}$$

or

$$M_5(y) = M_5(0) + 2\times 6^{\frac{1}{2}}SHKy. \tag{8'}$$

Finally, the integration of system (3) is reduced to the study of a single differential equation:

$$dN_1/dy = (2^{\frac{1}{2}}+3^{\frac{1}{2}})S(N_2N_3-N_1N_4)+\tfrac{1}{2}\times 6^{\frac{1}{2}}S(N_2M_4+N_3M_2-N_1M_1-N_1M_5). \tag{9}$$

† This result is general, for if the densities depend only on the variable $\xi = \alpha x+\beta y+\gamma z$ (α, β and γ being the components of a unit vector $\mathbf{v}$), the flux $\sum_i N_i\,\mathbf{u}_i.\mathbf{v}$ through a surface normal to $\mathbf{v}$ has as derivative

$$\sum_i (dN_i/d\xi)\mathbf{u}_i.\mathbf{v} = \sum_i (\partial N_i/\partial t+\mathbf{u}_i.\nabla N_i),$$

which is equal, by virtue of the kinetic equations, to the sum of all the collision terms; this sum is evidently zero, since to every collision there corresponds an inverse collision.

The second term is a function of N_1 and y only. When the constant H is zero, i.e. when $M_2(0) = M_5(0)$, the densities M_2 and M_5 have equal constant values and the integration of the differential equation (9) is reduced to a quadrature:

$$dN_1/dy = (2^{\frac{1}{2}}+3^{\frac{1}{2}})S(K-\tfrac{1}{2}M_2)(N_3-N_1) \\ +\tfrac{1}{2}\times 6^{\frac{1}{2}}SM_2(N_3-N_1)\{1+(K-\tfrac{1}{2}M_2)(N_1+N_3)^{-\frac{1}{2}}(N_2+N_4)^{-\frac{1}{2}}\} \quad (10)$$

with
$$\left.\begin{aligned} N_1+N_3 &= N_3(0)-N_1(0)+2N_1, \\ N_2+N_4 &= N_4(0)-N_2(0)+2N_2 = N_1(0)-N_3(0)+2K-M_2-2N_1. \end{aligned}\right\} \quad (11)$$

3. Calculation of densities

To compute the integral defined by (10), we introduce the non-dimensional quantities

$$n_i = N_i/K, \quad m_i = M_i/K, \quad \tilde{y} = \tfrac{1}{2}\times 6^{\frac{1}{2}}SKy, \quad (12)$$

and we put $\theta = 2^{\frac{1}{2}}+2/3^{\frac{1}{2}}$. The values of the non-dimensional densities for $\tilde{y} = 0$ are denoted by $\overline{n}_i$ and $\overline{m}_i$. With the assumptions made above, all straight lines with equations x = constant, $y = 0$ are axes of symmetry for the flow, so that we have

$$n_3(y) = n_2(-y), \quad n_4(y) = n_1(-y);$$

in particular, $\overline{n}_3 = \overline{n}_2$ and $\overline{n}_4 = \overline{n}_1$, so that

$$\overline{n}_1+\overline{n}_3+\tfrac{1}{2}\overline{m}_2 = 1. \quad (13)$$

All the densities are expressed in terms of the density n_1 by the formulae:

$$\left.\begin{aligned} &n_2 = \overline{n}_3-(n_1-\overline{n}_1), \quad n_3 = \overline{n}_3+(n_1-\overline{n}_1), \quad n_4 = \overline{n}_1-(n_1-\overline{n}_1), \\ &m_2 = m_5 = m_3 = m_6 = \overline{m}_2 = 2(1-\overline{n}_1-\overline{n}_3), \\ &m_1 = \overline{m}_2(n_2+n_4)^{\frac{1}{2}}(n_1+n_3)^{-\frac{1}{2}}, \quad m_4 = \overline{m}_2(n_1+n_3)^{\frac{1}{2}}(n_2+n_4)^{-\frac{1}{2}}. \end{aligned}\right\} \quad (14)$$

These densities are all positive provided

$$\overline{n}_1+\overline{n}_3 \leqslant 1, \quad 0 \leqslant n_1 \leqslant 2\overline{n}_1. \quad (15)$$

The constant $\overline{m}_2$ must be greater than zero, which corresponds to the model with eight velocities considered by Broadwell, and smaller than 2, which corresponds to the model with six velocities. Equation (10) can be written as

$$dn_1/d\tilde{y} = \theta(\overline{n}_3+\overline{n}_1)(\overline{n}_3-\overline{n}_1) \\ +\overline{m}_2(\overline{n}_3-\overline{n}_1)\{1+(\overline{n}_3+\overline{n}_1)(n_1+n_3)^{-\frac{1}{2}}(n_2+n_4)^{-\frac{1}{2}}\}, \quad (16)$$

which may be integrated in the form

$$\theta(\overline{n}_3-\overline{n}_1)\tilde{y} = \frac{\sin\phi}{1+\alpha}+\frac{(1-\alpha)\phi}{(1+\alpha)^2}+\frac{(1-\alpha)^2}{2\alpha^{\frac{1}{2}}(1+\alpha)^2}\log\frac{\alpha^{\frac{1}{2}}+\tan\frac{1}{2}\phi}{\alpha^{\frac{1}{2}}-\tan\frac{1}{2}\phi}, \quad (17)$$

where

$$\sin\phi = 4(n_1-\overline{n}_1)(2-\overline{m}_2)^{-1}, \quad \alpha = 1+2(3\times 2^{\frac{1}{2}}-2\times 3^{\frac{1}{2}})\overline{m}_2(2-\overline{m}_2)^{-1}. \quad (18)$$

The product $(\overline{n}_3-\overline{n}_1)\tilde{y}$ depends uniquely on the variable $n_1-\overline{n}_1$ and on the

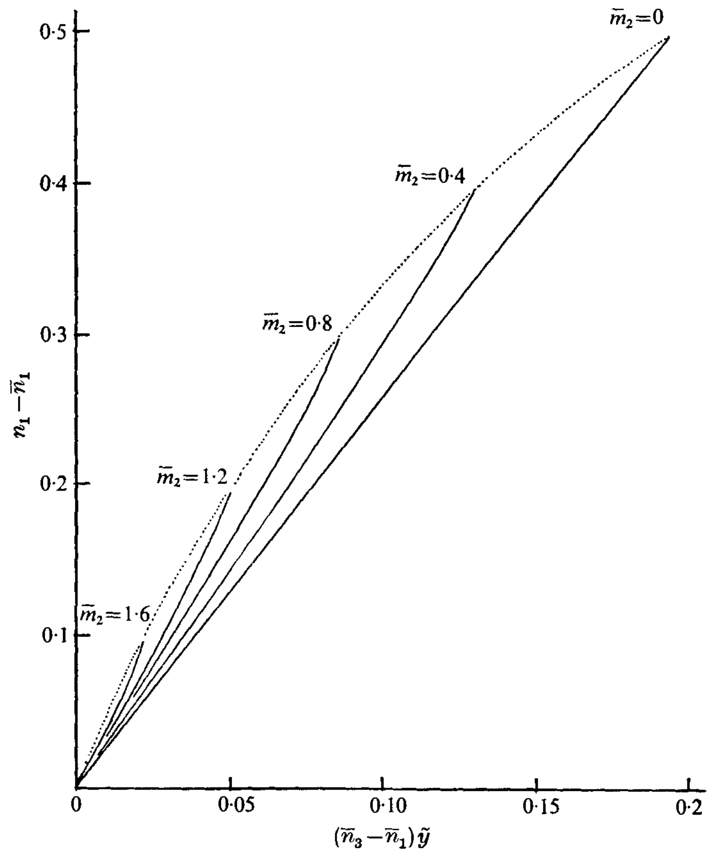

FIGURE 3. Values of the function $n_1(\tilde{y})$.

parameter $\bar{m}_2$. The function $n_1-\bar{n}_1$ is represented in figure 3. For $\bar{m}_2 = 0$, we have again the model with eight velocities, for which the function $n_1(\tilde{y})$ is linear. The maximum of $\tilde{y}$ is reached for $\phi = \frac{1}{2}\pi$.

When the densities have been computed, we can obtain the values of the macroscopic variables, i.e. the total number density n, the mean velocity $\mathbf{u}$ (with components u, v, w) and the temperature T. The density n can be obtained explicitly, in the symmetrical case ($H = 0$), by noticing that the densities m_2, m_3 and m_5 all have the same constant value $\bar{m}_2 = 2(1-\bar{n}_1-\bar{n}_3)$. We then have

$$\left.\begin{aligned} m_1 &= \bar{m}_2(n_2+n_4)^{\frac{1}{2}}(n_1+n_3)^{-\frac{1}{2}}, \quad m_4 = \bar{m}_2(n_1+n_3)^{\frac{1}{2}}(n_2+n_4)^{-\frac{1}{2}} \\ n_1+n_3 &= 1-\tfrac{1}{2}\bar{m}_2+2(n_1-\bar{n}_1), \quad n_2+n_4 = 1-\tfrac{1}{2}\bar{m}_2-2(n_1-\bar{n}_1), \end{aligned}\right\} \tag{19}$$

and

$$n = 8-(2-\bar{m}_2)\{2-\bar{m}_2(n_1+n_3)^{-\frac{1}{2}}(n_2+n_4)^{-\frac{1}{2}}\}. \tag{20}$$

The mean velocity is calculated from the flux

$$n\mathbf{u} = \sum_{i=1}^{8} n_i\mathbf{u}_i + \sum_{j=1}^{6} m_j\mathbf{v}_j,$$

giving

$$\left.\begin{aligned} nu &= -4c(n_1-\bar{n}_1)\{2+\bar{m}_2(n_1+n_3)^{-\frac{1}{2}}(n_2+n_4)^{-\frac{1}{2}}\}, \\ nv &= nw = 0. \end{aligned}\right\} \tag{21}$$

Finally, denoting by μ the mass of molecules and by k the Boltzmann constant, we obtain the expression for the temperature,

$$knT = \frac{\mu}{3}\left\{\sum_{i=1}^{8} n_i(\mathbf{u}_i - \mathbf{u})^2 + \sum_{j=1}^{6} m_j(\mathbf{v}_j - \mathbf{u})^2\right\}$$

and so
$$kT = \tfrac{1}{3}\mu c^2\{1-(u/c)^2 + 4n^{-1}(2-\overline{m}_2)\}. \tag{22}$$

When the gas is at rest in a Maxwellian state, we have $n = 4(1+\overline{m}_2)$ and $(1+\overline{m}_2)\,kT = \mu c^2$; the temperature then lies between the extreme values

$$T_m = \tfrac{1}{3}k^{-1}\mu c^2 \quad \text{for} \quad \overline{m}_2 = 2 \quad \text{(gas with six velocities)},$$

$$T_M = k^{-1}\mu c^2 \quad \text{for} \quad \overline{m}_2 = 0 \quad \text{(gas with eight velocities)}.$$

Note that
$$T/T_M = \tfrac{1}{3}\{1-(u/c)^2 + 4n^{-1}(2-\overline{m}_2)\}. \tag{23}$$

The macroscopic variables depend on the parameter $\overline{m}_2$ and on the variable $n_1 - \overline{n}_1$.

4. Determination of the flow

The solution of the kinetic equations given in the previous section depends on four parameters: $\overline{n}_1$, $\overline{n}_2$, $\overline{m}_2$ and K. The parameter K is proportional to the density N_0 at rest, for we have

$$N_0 = \sum_{i=1}^{8} N_1 + \sum_{j=1}^{6} M_j = 4K(1+\overline{m}_2), \tag{24}$$

a formula which gives K when $\overline{m}_2$ is known.

To determine the other three parameters, it is appropriate to write down the boundary conditions on the walls ($y = \pm d$), on which we have

$$\tilde{y} = \pm\tilde{y}_w = \pm\tfrac{1}{2}\times 6^{\frac{1}{2}}SKd = \pm\tfrac{1}{8}\times 6^{\frac{1}{2}}K_n^{-1}(1+\overline{m}_2)^{-1}, \tag{25}$$

where
$$K_n = (SN_0d)^{-1} \tag{26}$$

is the Knudsen number. We shall adopt as boundary conditions on the walls the laws of diffuse reflexion. The molecules reflected by the lower wall ($y = -d$) are those whose y component of velocity is positive, i.e. those with velocities $\mathbf{u}_1$, $\mathbf{u}_2$, $\mathbf{u}_5$, $\mathbf{u}_6$ and $\mathbf{v}_2$. The molecules reflected by the upper wall ($y = d$) are those whose y component of velocity is negative, i.e. those with velocities $\mathbf{u}_3$, $\mathbf{u}_4$, $\mathbf{u}_7$, $\mathbf{u}_8$ and $\mathbf{v}_5$.

The laws of diffuse reflexion state first that on each wall the projection on the wall of the mean velocity of the reflected molecules is equal to the velocity of the wall. We denote by n^+ the density of the molecules reflected by the lower wall, by $\mathbf{u}^+$ (components u^+, v^+, w^+) their mean velocity, and by n^- the density of the molecules reflected by the upper wall and by $\mathbf{u}^-$ (components u^-, v^-, v^-) their mean velocity. We have then

$$n^+ = 2(n_1+n_2)+m_2 = 2, \quad n^- = 2(n_3+n_4)+m_5 = 2 \tag{27}$$

and
$$\left.\begin{aligned} n^+\mathbf{u}^+ &= n_1\mathbf{u}_1 + n_2\mathbf{u}_2 + n_5\mathbf{u}_5 + n_6\mathbf{u}_6 + m_2\mathbf{v}_2,\\ n^-\mathbf{u}^- &= n_3\mathbf{u}_3 + n_4\mathbf{u}_4 + n_7\mathbf{u}_7 + n_8\mathbf{u}_8 + m_5\mathbf{v}_5 \end{aligned}\right\} \tag{28}$$

and therefore

$$\left.\begin{aligned} u^+ &= c(n_2-n_1) = c(1-\tfrac{1}{2}\overline{m}_2-2n_1), \quad v^+ = c, \quad w^+ = 0, \\ u^- &= c(n_4-n_3) = -c\{1-\tfrac{1}{2}\overline{m}^2-2(2\overline{n}_1-n_1)\}, \quad v^- = -c, \quad w^- = 0. \end{aligned}\right\} \tag{29}$$

The boundary conditions then give

$$U_w/c = \begin{cases} u^+/c = 1-\tfrac{1}{2}\overline{m}_2-2n_1 \quad \text{for } y = -d, & (30a) \\ -u^-/c = 1-\tfrac{1}{2}\overline{m}_2-2(2\overline{n}_1-n_1) \quad \text{for } y = d. & (30b) \end{cases}$$

The laws of diffuse reflexion next state that on each wall the temperature of the reflected molecules, defined by regarding the velocity of the wall as the mean velocity, is equal to the temperature of the wall; this gives on the lower and upper walls respectively

$$\left.\begin{aligned} kn^+T_w &= \tfrac{1}{3}\mu\{2n_1(\mathbf{u}_1-U_w)^2+2n_2(\mathbf{u}_2-U_w)^2+m_2(\mathbf{v}_2-U_w)^2\}, \\ kn^-T_w &= \tfrac{1}{3}\mu\{2n_3(\mathbf{u}_3+U_w)^2+2n_4(\mathbf{u}_4+U_w)^2+m_5(\mathbf{v}_5+U_w)^2\}, \end{aligned}\right\} \tag{31}$$

or

$$T_w/T_M = \tfrac{1}{3}\{1-(U_w/c)^2+2-\overline{m}_2\}.$$

The boundary conditions on the walls thus determine $\overline{m}_2$,

$$\overline{m}_2 = 3-3T_w/T_M-(U_w/c)^2, \tag{32}$$

and also the (equal) values of n_1 on the lower wall and $2\overline{n}_1-n_1$ on the upper wall:

$$4(n_1)_{y=-d} = 3T_w/T_M+(1-U_w/c)^2-2 = 4(2\overline{n}_1-n_1)_{y=d}. \tag{33}$$

As the velocity U_w is assumed positive, we have

$$0 \leqslant (n_1)_{y=-d} \leqslant \tfrac{1}{4}(2-\overline{m}_2), \quad 0 \leqslant (2\overline{n}_1-n_1)_{y=d} \leqslant \tfrac{1}{4}(2-\overline{m}_2). \tag{34}$$

In figure 4 we show on the plane with U_w/c as abscissa and T_w/T_M as ordinate the curves on which $\overline{m}_2$ = constant and the curves (dotted) on which $(n_1)_{y=-d}$ (or $(2\overline{n}_1-n_1)_{y=d}$) = constant. The fact that these quantities are positive leads to the inequalities

$$2-(1-U_w/c)^2 \leqslant 3T_w/T_M \leqslant 3-(U_w/c)^2, \tag{35}$$

which can be satisfied only if the velocity of the walls is less than c.

To complete the determination of the flow, we must still obtain the parameters $\overline{n}_1$ and $\overline{n}_3$, which satisfy the relation

$$2(\overline{n}_1+\overline{n}_3) = 3T_w/T_M+(U_w/c)^2-1.$$

When those parameters are known, we know by virtue of (30a) the value of $n_1-\overline{n}_1$ on the lower wall ($y = -d$), and the formula (17) gives the value $-\tilde{y}_w$ of $\tilde{y}$ on the wall. As we have also

$$\tilde{y}_w = +\tfrac{1}{2}\times 6^{\frac{1}{2}}SKd = \tfrac{1}{8}\times 6^{\frac{1}{2}}SN_0d(1+\overline{m}_2)^{-1}, \tag{36}$$

we deduce the value of the Knudsen number K_n as a function of U_w/c, T_w/T_M and $\overline{n}_1$; this is given in table 1. As, in practice, the Knudsen number is given, table 1 (by interpolation) and the formulae (30) entirely determine the flow. An example corresponding to $T_w/T_M = \frac{2}{3}$ and $U_w/c = 0{\cdot}8$ and to three values of the Knudsen number is shown in figure 5. For $K_n = 0$, it is quite clear that the velocity

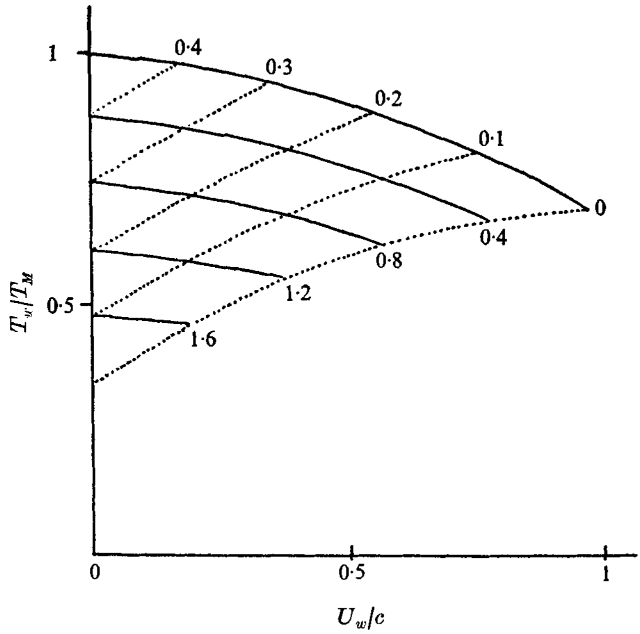

FIGURE 4. Curves $\overline{m}_2$ = constant (solid lines) and $(n_1)_{y=-d}$ = constant (dotted lines).

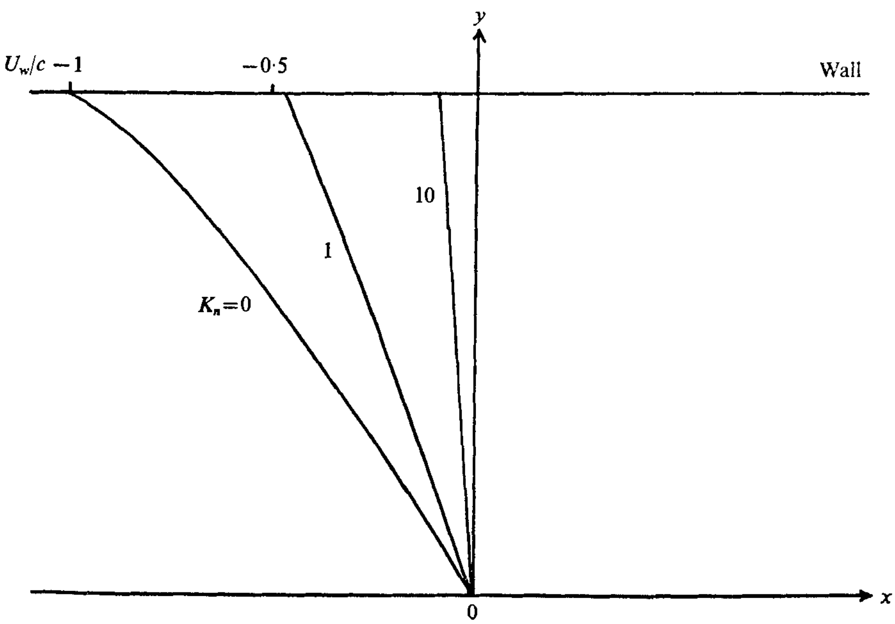

FIGURE 5. Velocity profiles (model with fourteen velocities). $T_w/T_M = \frac{2}{3}$, $U_w/c = 0{\cdot}8$.

$\bar{n}_1 - n_1$	T_w/T_M: $\frac{2}{3}$	0·733	0·786	0·833	0·830	0·933	0·946	0·986
				$U_w/c = 0{\cdot}2$				
0·00	∞	∞		∞		∞		∞
0·01	9·156	9·748		10·940		12·749		14·158
0·02	4·070	4·333		4·863		5·666		6·293
0·03	2·375	2·528		2·837		3·306		3·671
0·03	1·528	1·626		1·824		2·125		2·360
0·05	1·019	1·084		1·216		1·417		1·573
0·06	0·680	0·723		0·811		0·945		1·049
0·07	0·438	0·465		0·521		0·607		0·674
0·08	0·256	0·272		0·304		0·354		0·393
0·09	0·114	0·121		0·135		0·157		0·175
0·10	0·000	0·000		0·000		0·000		0·000
				$U_w/c = 0{\cdot}4$				
0·00	∞	∞		∞		∞	∞	
0·02	9·497	10·173		11·567		13·765	14·158	
0·04	4·224	4·523		5·142		6·118	6·293	
0·06	2·466	2·640		3·000		3·569	3·671	
0·08	1·588	1·699		1·929		2·294	2·360	
0·10	1·061	1·134		1·287		1·530	1·573	
0·12	0·709	0·757		0·858		1·020	1·049	
0·14	0·457	0·488		0·552		0·656	0·674	
0·16	0·268	0·285		0·322		0·382	0·393	
0·18	0·120	0·127		0·143		0·170	0·175	
0·20	0·000	0·000		0·000		0·000	0·000	
				$U_w/c = 0{\cdot}6$				
0·00	∞	∞		∞	∞			
0·03	10·175	11·039		12·904	14·158			
0·06	4·526	4·908		5·736	6·293			
0·09	2·644	2·865		3·347	3·671			
0·12	1·703	1·844		2·152	2·360			
0·15	1·139	1·231		1·435	1·573			
0·18	0·762	0·822		0·957	1·049			
0·21	0·492	0·530		0·616	0·674			
0·24	0·289	0·310		0·359	0·393			
0·27	0·130	0·138		0·160	0·175			
0·30	0·000	0·000		0·000	0·000			
				$U_w/c = 0{\cdot}8$				
0·00	∞	∞	∞					
0·04	11·457	12·750	14·158					
0·08	5·095	5·668	6·293					
0·12	2·975	3·308	3·671					
0·16	1·916	2·128	2·360					
0·20	1·280	1·419	1·573					
0·24	0·856	0·947	1·049					
0·28	0·552	0·610	0·674					
0·32	0·324	0·356	0·393					
0·36	0·145	0·159	0·175					
0·40	0·000	0·000	0·000					

TABLE 1. Knudsen number

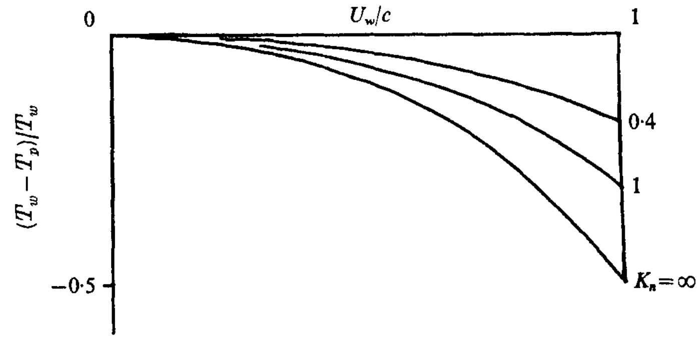

FIGURE 6. Temperature slip coefficient (Broadwell gas).

profile is curved; this is an improvement on the Broadwell model (with eight velocities), for which the velocity profile had the linear form

$$u/U_w = -(y/d)\{1+2K_n(2^{\frac{1}{2}}+3^{\frac{1}{2}})^{-1}\}^{-1}, \tag{37}$$

and the temperature profile has the parabolic form

$$T/T_M = 1-\tfrac{1}{3}(u/c)^2. \tag{38}$$

A particular point of interest in the solution of the Couette flow problem is the study of slip coefficients, corresponding to slip of velocity and temperature. In the case of a Broadwell gas, we obtain for those two coefficients the expressions

$$\alpha = (U_w - U_p)/U_w = 1-\{1+2K_n(2^{\frac{1}{2}}+3^{\frac{1}{2}})^{-1}\}^{-1} \tag{39}$$

and

$$\beta = (T_w - T_p)/T_w = 1-\{1-\tfrac{1}{3}(U_p/c)^2\}/\{1-\tfrac{1}{3}(U_w/c)^2\}. \tag{40}$$

U_p and T_p denote respectively the velocity and the temperature of the fluid on the wall. The coefficient α depends only on the Knudsen number; the coefficient β depends on the Knudsen number and on the ratio U_w/c; its values are indicated in figure 6. In the model with fourteen velocities the two slip coefficients α and β depend on the three parameters U_w/c, T_w/T_M and K_n. The coefficient of slip velocity is represented in figure 7 (*a*) for $T_w/T_M = \frac{2}{3}$ and in figure 7 (*b*) for $T_w/T_M = 0{\cdot}8$. For all values of T_w other than $\frac{2}{3}T_M$ the velocity slip coefficient does not go to zero in the continuous flow limit $K_n \to 0$. This result, in contradiction with experiments, is due to the velocities $\mathbf{v}_1$, $\mathbf{v}_3$, $\mathbf{v}_4$ and $\mathbf{v}_6$, which are tangential to the wall (while in reality velocities tangential to the wall constitute a set of measure zero).

5. Model with ten velocities

It is possible to remove the four velocities tangential to the wall, for if we assume that the molecules can have as velocity only one of the vectors $\mathbf{u}_i$ and one of the vectors $\mathbf{v}_2$ or $\mathbf{v}_5$, we obtain a model with ten velocities, which allows of course the same collisions as before between the molecules of velocity $\mathbf{u}_i$ and also the four mixed collisions

$$(\mathbf{u}_3, \mathbf{v}_2) \leftrightarrow (\mathbf{u}_1, \mathbf{v}_5), \quad (\mathbf{u}_7, \mathbf{v}_2) \leftrightarrow (\mathbf{u}_5, \mathbf{v}_5),$$

$$(\mathbf{u}_4, \mathbf{v}_2) \leftrightarrow (\mathbf{u}_2, \mathbf{v}_5), \quad (\mathbf{u}_8, \mathbf{v}_2) \leftrightarrow (\mathbf{u}_6, \mathbf{v}_5).$$

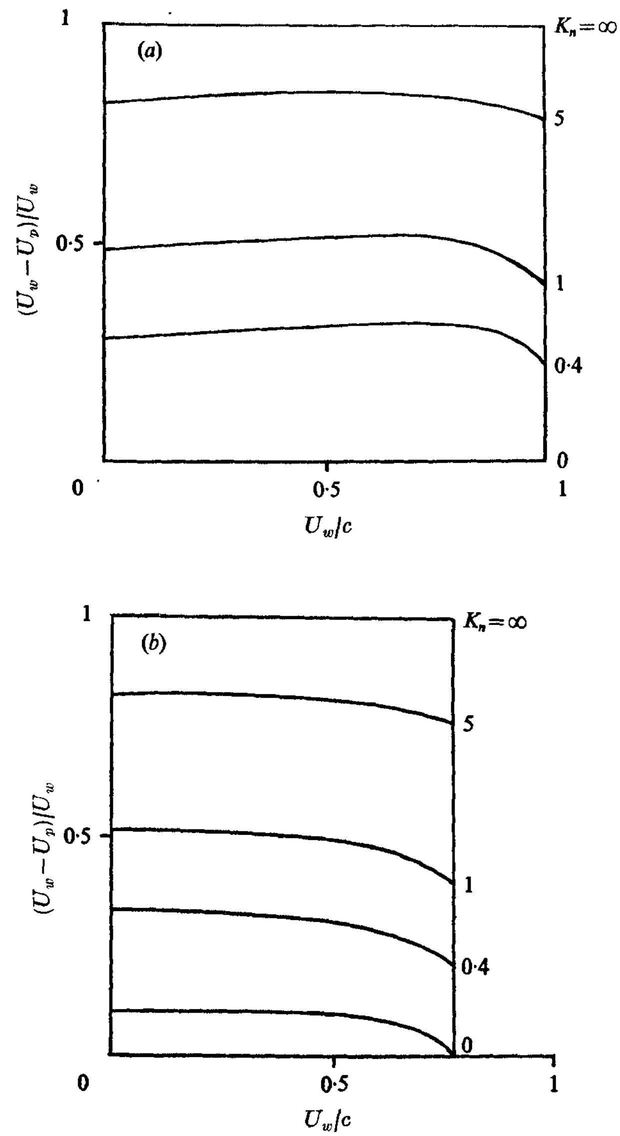

FIGURE 7. Velocity slip coefficient for (a) $T_w = \frac{2}{3}k^{-1}\mu c^2$ and (b) $T_w = \frac{4}{5}k^{-1}\mu c^2$ (model with fourteen velocities).

Assuming always that the densities depend only on y, and that $N_{i+4} = N_i$, the kinetic equations must now be replaced by

$$\left.\begin{aligned} dN_1/dy = dN_3/dy &= (2^{\frac{1}{2}}+3^{\frac{1}{2}})S(N_2N_3 - N_1N_4) + \tfrac{1}{2}\times 6^{\frac{1}{2}}S(N_3M_2 - N_1M_5),\\ dN_2/dy = dN_4/dy &= (2^{\frac{1}{2}}+3^{\frac{1}{2}})S(N_1N_4 - N_2N_3) + \tfrac{1}{2}\times 6^{\frac{1}{2}}S(N_4M_2 - N_2M_5),\\ dM_2/dy = dM_5/dy &= 6^{\frac{1}{2}}S\{(N_1+N_2)M_5 - (N_3+N_4)M_2\}. \end{aligned}\right\} \quad (41)$$

The difference $M_5 - M_2$ is a constant, zero when the two walls are at the same

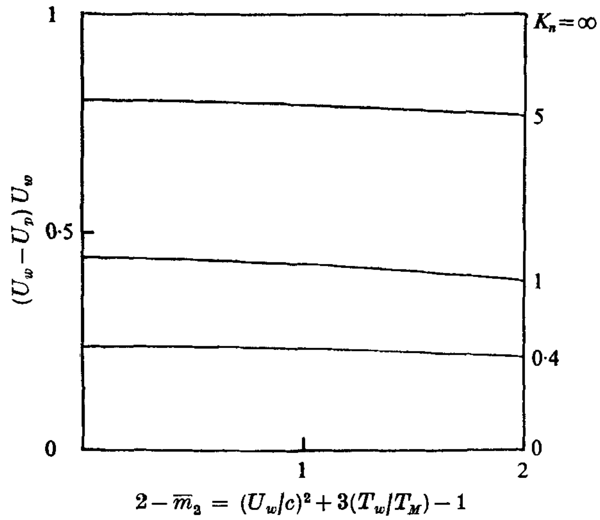

FIGURE 8. Velocity slip coefficient (model with ten velocities).

temperature T_w, as we shall suppose. With the same notation as in § 3, (41) can be integrated to give

$$\left.\begin{aligned} n_1 &= \overline{n}_1 + \alpha\tilde{y}, \quad n_3 = \overline{n}_3 + \alpha\tilde{y},\\ n_2 &= \overline{n}_3 - \alpha\tilde{y}, \quad n_4 = \overline{n}_1 - \alpha\tilde{y},\\ \alpha &= \{\overline{m}_2 + (2-\overline{m}_2)(2^{\frac{1}{2}} + 3^{\frac{1}{2}})/6^{\frac{1}{2}}\}(\overline{n}_3 - \overline{n}_1), \end{aligned}\right\} \tag{42}$$

where $\overline{m}_2$ has the constant value $\overline{m}_2 = 2(1-\overline{n}_1-\overline{n}_3)$. The macroscopic variables can be expressed in the simple forms

$$\left.\begin{aligned} n &= \sum_{i=1}^{8} N_i + M_2 + M_5 = 4K \qquad \text{(density)},\\ u &= -2c\alpha\tilde{y}, \quad v = w = 0 \qquad \text{(velocity)},\\ kT &= \tfrac{1}{3}\mu c^2\{1-(u/c)^2 + 2 - \overline{m}_2\} \qquad \text{(temperature)}. \end{aligned}\right\} \tag{43}$$

The boundary conditions on the walls are

$$\left.\begin{aligned} U_w &= c\{\overline{n}_1 - \overline{n}_3 - 2\alpha\tilde{y}_w\},\\ kT_w &= \tfrac{1}{3}\mu c^2\{1-(U_w/c)^2 + 2 - \overline{m}_2\},\\ \tilde{y}_w &= \tfrac{1}{8}\times 6^{\frac{1}{2}} K_n^{-1}. \end{aligned}\right\} \tag{44}$$

We deduce the values of the velocity and temperature slip coefficients

$$(U_w - U_p)/U_w = 1 - \{1 + [\tfrac{1}{2}(2^{\frac{1}{2}} + 3^{\frac{1}{2}}) + \tfrac{1}{4}(6^{\frac{1}{2}} - 3^{\frac{1}{2}} - 2^{\frac{1}{2}})\overline{m}_2]^{-1}\}^{-1}, \tag{45}$$

and

$$(T_w - T_p)/T_w = 1 - \tfrac{1}{3}(3 - \overline{m}_2 - 3\alpha^2/8K_n^2)\, T_w/T_M. \tag{46}$$

The velocity slip coefficient is shown in figure 8. This coefficient, always positive, is a decreasing function of the temperature and velocity of the walls and an increasing function of the Knudsen number, being zero in the case of continuous

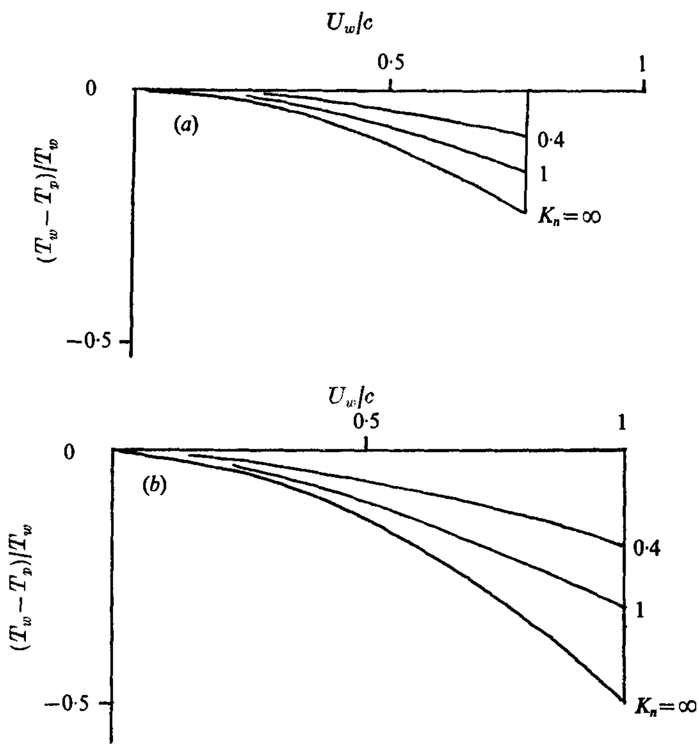

FIGURE 9. Temperature slip coefficient for (a) $T_w/T_M = 0{\cdot}8$ and (b) $T_w/T_M = \frac{2}{3}$ (model with ten velocities).

flow ($K_n = 0$). The temperature slip coefficient is shown in figure 9(a) for $T_w/T_M = 0{\cdot}8$ and in figure 9(b) for $T_w/T_M = \frac{2}{3}$. This coefficient, always negative, is a decreasing function of the Knudsen number, zero in the case of continuous flow ($K_n = 0$).

6. Conclusions

The above results are interesting from several points of view. They allow us to compare for the same physical problem the value of different models with discrete velocity distributions. The very simple model of Broadwell led to formulae which (from a qualitative point of view) were surprisingly accurate; the introduction of velocities of different moduli has, however, allowed the introduction of temperature as an independent macroscopic variable, and it is noteworthy that the model with ten velocities gives better results than the model with fourteen velocities, because of the absence of velocities parallel to the 'preferred' directions of the problem (direction of the walls, direction of the flow). This is probably general: the same phenomenon will probably emerge in all problems of kinetic theory solved by discretization of velocity space. The different formulae obtained for the model with ten velocities lead to results which are qualitatively in complete conformity with experimental results as regards both the velocity slip coefficient and the temperature slip coefficient. One may hope that a further increase in the number of vectors available to the velocities may lead in the future to quantitative conformity also.

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